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Exponential Decay of the Power Spectrum of Turbulence

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The analyticity on a strip of the solutions of Navier–Stokes equations in 2D is shown to explain the observed fast decay of the frequency power spectrum of the turbulent velocity field. Some subtleties in the application of the Wiener–Khinchine method to turbulence are resolved by showing that the frequency power spectrum of turbulent velocities is in fact a measure exponentially decaying for frequency $\rightarrow \pm\infty$. Our approach also shows that the conventional procedures used in analyzing data in turbulence experiments are valid even in the absence of the ergodic property in the flow.

KEY WORDS: Turbulence; temporal velocity fluctuations; statistical power spectra; analyticity; ergodicity; invariant probability measure.

1. INTRODUCTION

The empirical basis for many theories of fluid turbulence is the observed power spectrum characterizing the flow. In fact the spectrum as such is not measured directly. Rather, the behavior of the velocity at a point is observed as a function of time, and the Taylor hypothesis is used to relate the time sequence to the behavior of the velocity in space. A useful measurement is the two-time correlation of the turbulent velocity at a point in space. The Fourier transform of that correlation then yields the frequency spectrum of the velocity, say $P(\omega)$, at a given point in the turbulent flow. In the course of those measurements it was observed that $P(\omega)$ drops off very rapidly at high frequencies.[11] How fast is still a matter

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of debate.[1] We show that $P(\omega)$ decays at least as fast as $\exp(-|\omega|/|\omega_d|)$. Because such measurements are among the few quantities which can be obtained from experiments, it is all the more important to ensure that the operations on the data make mathematical sense, and therefore it is useful to establish the ensuing theoretical considerations on a sound mathematical basis. Here we show that this behavior can be related to the time analyticity on a strip with finite width of the solutions of 2D Navier–Stokes equations.

As a step to showing the connection between the analyticity property and the high-frequency behavior of the power spectrum, it is necessary to characterize precisely the mathematical objects we deal with, namely the time-correlation function and its Fourier transform. Although on the surface it would appear that the well-known apparatus of the Wiener–Khintchine theory[6] connecting the time behavior of the correlation function and the frequency power spectrum suffices, there are some subtle points not previously discussed in this context. Their clarification leads to a useful result which shows that the standard operations may be valid even in the absence of ergodicity of the flow. Also our approach shows that the conventional Wiener–Khintchine method used in turbulent velocity analysis is justified, provided one works in an adequately constructed mathematical framework.

We note that the mathematical apparatus used here is more or less well known, but we believe that its application here is new and illuminating.

In addition to the important mathematical consequences of the analyticity property, analyticity can also be interpreted physically. Indeed, the analyticity property is equivalent to the fact that at no time does the momentum in the turbulent flow become infinite. The experimental evidence suggests that this property also holds for 3D flows. If this were true, then our rigorous approach would also be valid for 3D flows.

Section 2 introduces our notation and some of the needed properties of the Navier–Stokes equations in two dimensions. In Section 3 and subsequently in Section 5 we show how the pointwise spectrum at high frequencies depends on the width of the analyticity strip in the complex time domain. We also show that the pointwise properties of the power spectrum $P(\omega)$ include the possibility that this spectrum may not be an ordinary function, but a distribution. That possibility raises many practical difficulties, and to mitigate them, in Section 4 we examine the power spectrum as an object with well-understood statistical properties, and in particular as based on a process with an invariant probability measure. As a result it follows that $P(\omega)$ as a distribution is a measure. Section 5 addresses the existence of an invariant measure such that the necessary time averages exist at least in the sense of the Banach limit, eventually ensuring the existence and meaningfulness of the power spectrum as a Fourier transform of the two-time correlation function. Some comments about the corresponding situation in three-dimensional turbulence are contained in Section 6, and conclusions are summarized in Section 7.

2. PRELIMINARIES ON THE NAVIER–STOKES EQUATION

2.1. We consider the 2D Navier–Stokes equations with either periodic boundary conditions or no-slip boundary conditions. To be precise we consider $\Omega = [0, L]^2$ in the first case and Ω a connected bounded domain with a C^2-boundary in the second case. We set

$$V = \Big\{ u = \sum_{k \in \mathbb{Z}^2} a_k e^{(2\pi i/L) k \cdot x}: \quad a_k \in \mathbb{R}^2,\ a_k = 0 \text{ for } k \cdot k \text{ large enough},\ a_0 = 0,\ \nabla \cdot u = 0 \Big\}$$

in the periodic case and

$$V = \{ u \in C_0^\infty(\Omega)^2 : \nabla \cdot u = 0 \}$$

in the no-slip case. We denote by H, respectively V, the closure of V in $L^2(\Omega)^2$, respectively in $H^1(\Omega)^2$. [We recall that $H^l(\Omega)$, $l = 1, 2,\ldots$, is the space of all $\varphi \in L^2(\Omega)$ such that their derivatives (as distributions) are in $L^2(\Omega)$ up to order l.] In H and V we consider the scalar products

$$(u, v) = \int_\Omega u(x) \cdot v(x)\, dx, \qquad ((u, v)) = \int_\Omega \sum_{i=1}^{2} \frac{\partial u}{\partial x_i}(x) \cdot \frac{\partial v}{\partial x_i}(x)\, dx$$

respectively. The corresponding norms are denoted by

$$|u| = (u, u)^{1/2} \quad (u \in H), \qquad \|v\| = ((v, v))^{1/2} \quad (v \in V)$$

The orthogonal projection of $L^2(\Omega)^2$ onto H will be denoted by P. The phase space of the 2D Navier–Stokes equations with the boundary condition considered above is H. To obtain the dynamical system of the 2D Navier–Stokes equations in H, one applies (following Leray[7]) the projection P to the linear momentum equations of the Navier–Stokes equations, obtaining the differential equation

$$\frac{du}{dt} + \nu A u + B(u, u) = f \tag{2.1.1}$$

in H. The operators A and B are defined by

$$A = -P\Delta \tag{2.1.2}$$

on $V \cap H^2(\Omega)^2$, where $\Delta = \nabla^2$ is the Laplace operator, and

$$B(u, v) = P[(u \cdot \nabla) v], \qquad u, v \in \mathscr{D}(A) = V \cap H^2(\Omega)^2 \tag{2.1.3}$$

The right-hand side f represents the divergence-free component of the body forces, which for convenience will be assumed in H. Of course, ν is the kinematic viscosity of the fluid.

We recall that solutions of (2.1.1) are known to exist for all time $t \geqslant 0$. That is, for every initial datum $u_0 \in H$ there exists a unique continuous function u: $[0, \infty) \mapsto H$ such that:

(i) $u(0) = u_0$.

(ii) (2.1.1) is satisfied for $t > 0$, that is, $u(t) \in \mathscr{D}(A)$ for $t > 0$ and du/dt exists in H for $t > 0$ and satisfies (2.1.1).

In fact, if $u_0 \in V$ [or $u_0 \in \mathscr{D}(A)$], then $u(t)$ is continuous in V [resp. in $\mathscr{D}(A)$] on $[0, \infty)$; recall that the norm of V is $\|\cdot\|$ and the norm of $\mathscr{D}(A)$ is $|A\cdot|$. The map $u_0 \mapsto u(t)$ defined for all $t \geqslant 0$ and $u_0 \in H$ is denoted by $S(t)$. The dynamical system associated to the 2D Navier–Stokes equations with either one of the boundary conditions specified above is the nonlinear semigroup $\{S(t)\}_{t \geqslant 0}$.

2.2. One of the most important properties of $\{S(t)\}_{t \geqslant 0}$ is dissipativity, that is, the existence of an absorbing compact set B_a in H for $\{S(t)\}_{t \geqslant 0}$. This means that for every bounded set B in H there exists a time $t_0 = t_0(B)$ such that $S(t)\mathscr{B} \subset B_a$ for all $t \geqslant t_0$. In fact, B_a can be chosen to be a closed ball in V, i.e.,

$$B_a = \{u \in V\colon \|u\| \leqslant r_1\} \tag{2.2.1}$$

The number r_1 must be chosen to be sufficiently large. In the periodic case the inequality

$$r_1 > |f|/\nu\lambda_1^{1/2} = (G\,|f|)^{1/2} \tag{2.2.2}$$

will suffice, while in the no-slip case one must require

$$r_1 > 2e^{c_1 G^4}(G\,|f|)^{1/2} \tag{2.2.3}$$

Here G is the dimensionless generalized Grashoff number, namely

$$G = |f|/\nu^2\lambda_1 \tag{2.2.4}$$

and c_1 (as well as $c_2, c_3,\ldots$ in the sequel) denotes an absolute (dimensionless) constant in general of the order of unity. It is still an open question whether the exponential in (2.2.3) is superfluous or not.

We also note that the absorbing set B_a can be chosen to be a closed ball in $\mathscr{D}(A)$; in this case the estimates of its radius in terms of G are more complicated than (2.2.2)–(2.2.3). It will, however, be more convenient in the sequel to consider an absorbing set B_a which is a closed ball in $\mathscr{D}(A)$.

2.3. The main feature of a dissipative dynamical system is its *global attractor* $\mathscr{A}$ defined by

$$\mathscr{A} = \bigcap_{t \geqslant 0} S(t)\, B_a \tag{2.3.1}$$

This set has the following properties:

(i) $\mathscr{A}$ is a nonempty compact subset of H.

(ii) For any bounded set B in H and any $\varepsilon > 0$ there exists $t_0 = t_0(\varepsilon, B)$ such that

$$S(t)\, B \subset \{u \in H\colon \operatorname{dist}_H(u, \mathscr{A}) < \varepsilon\}$$

for all $t \geqslant t_0$.

(iii) $\mathscr{A}$ is the smallest set enjoying properties (i), (ii).

(iv) $S(t)\,\mathscr{A} = \mathscr{A}$ for all $t \geqslant 0$ [in other words, the solution $u(t)$ of (2.1.1) satisfying $u(0) = u_0 \in \mathscr{A}$ exists for all real t and lies in $\mathscr{A}$].

(v) $\mathscr{A}$ is the set of all vectors $u_0 \in H$ such that the solution $u(t)$ of (2.1.1) satisfying $u(0) = u_0$ exists for all real t and is bounded on the whole $(-\infty, \infty)$.

(vi) $\mathscr{A}$ is compact even as a subset of $\mathscr{D}(A)$.

(vii) If the generalized Grashoff number G is less than a certain absolute constant c_2, then $\mathscr{A} = \{u_0\}$, where u_0 is a stationary solution of (2.1.1), i.e., $\nu A u_0 + B(u_0, u_0) = f$.

2.4. Another useful property of the global attractor is the global time analyticity of the solutions with initial data in $\mathscr{A}$. To be specific, let $H_{\mathbb{C}}$, $V_{\mathbb{C}}$, and $\mathscr{D}(A)_{\mathbb{C}}$ denote the complexifications of the spaces H, V, and $\mathscr{D}(A)$, respectively. Thus, for instance, $H_{\mathbb{C}} = \{u_1 + iu_2\colon u_1, u_2 \in H\}$. The operator A is extended to $\mathscr{D}(A)_{\mathbb{C}}$ by the formula $A(u_1 + iu_2) = Au_1 + iAu_2$, $u_1, u_2 \in \mathscr{D}(A)$. Working with complexifications amounts to considering vector fields whose components are allowed to be complex-valued.

The analyticity mentioned above is the following property. There exist $\delta_0, \beta_0 > 0$, depending only on $\mathscr{A}$ [that is, only on Eq. (2.1.1)] such that for every $u_0 \in \mathscr{A}$, the function $S(t)\,u_0$ extends from $[0, \infty)$ to a $\mathscr{D}_{\mathbb{C}}(A)$-valued analytic function $U(\zeta, u_0)$ on the whole band $\{\zeta \in \mathbb{C}\colon |\Im\zeta| \leqslant \delta_0\}$ satisfying also

$$|A\,U(\zeta, u_0)| \leqslant \beta_0 \qquad \text{for all} \quad \zeta \in \mathbb{C}, \quad |\Im\zeta| \leqslant \delta_0 \tag{2.4.1}$$

Note that for $\zeta = t$ real, $U(t, u_0)$ is precisely the solution $u(t)$ of (2.1.1) on the whole real time line satisfying $u(0) = u_0$; in particular,

$$U(t, u_0) = S(t)\,u_0 \qquad \text{for} \quad t \geqslant 0, \quad u_0 \in \mathscr{A} \tag{2.4.2}$$

For later use let us note that we can extend the restriction of the (non-linear) semigroup $\{S(t)\}_{t \geqslant 0}$ to $\mathscr{A}$ to a (nonlinear) group $\{S_{\mathscr{A}}(t)\}_{-\infty < t < \infty}$ by defining

$$S_{\mathscr{A}}(t)\,u_0 = \begin{cases} S(t)\,u_0 & \text{if} \quad t > 0 \\ U(t, u_0) & \text{if} \quad t \leqslant 0 \end{cases} \tag{2.4.3}$$

for $u_0 \in \mathscr{A}$. The quantities δ_0 and β_0 introduced above can be estimated explicitly. For instance, one can take

$$\delta_0 = (\nu\lambda_1)^{-1}\, c_3 G^{-1} (\log^+ G + 1)^{-1} \tag{2.4.4}$$

in the periodic case, and

$$\delta_0 = (\nu\lambda_1)\, c_3 e^{-c_1 G^4} (2G)^{-1}\, [\log(2G+1) + 1 + c_1 G^4]^{-1} \tag{2.4.5}$$

is the no-slip case, where c_3 is a new appropriate absolute constant. Also, β_0 can be taken equal to a moderate multiple of the radius of B_a in $\mathscr{D}(A)$.

All the statements above can be either found explicitly in ref. 2 or ref. 9, or can be easily proved by the techniques presented in those monographs.

2.5. *Remark.* An important consequence of the existence of compact absorbing subsets of $\mathscr{D}(A)$ is the fact that for any solution u of (2.1.1), the functions $(du/dt)(t, x_0)$, $x_0 \in \Omega$, are bounded for large t. In particular, $u(t, x_0)$ is uniformly continuous for large values of t. This fact will be used several times, particularly in Section 4.3 and Section 5.

3. THE POINTWISE POWER SPECTRUM

3.1. Let $g(t)$ be a bounded, measurable, real-valued function depending on the real variable $t \in (-\infty, \infty)$. The *power spectrum* $P(\omega)$ of $g(t)$ is usually defined by

$$P(\omega) = \lim_{T \to \infty} \frac{1}{T} \left| \int_0^T e^{-i\omega t} g(t)\, dt \right|^2 \tag{3.1.1}$$

provided that the limit exists. In order to increase the range of applicability of this concept, it is useful to view $P(\omega)$ as a distribution in the sense of L. Schwartz. We refer to ref. 4 for the facts related to distribution theory. Recall that distributions are linear functionals on the space $\mathscr{D}(\mathbb{R})$ of all compactly supported, infinitely differentiable functions on the real line, continuous in a certain topology. The action of a distribution P on a test function $\varphi \in \mathscr{D}(\mathbb{R})$ will be written as $\int_{-\infty}^{\infty} P(\omega)\, \varphi(\omega)\, d\omega$ even when P is not a function. A distribution P is said to be a positive measure if there exists a positive Borel measure m on the line such that $\int_{-\infty}^{\infty} P(\omega)\, \varphi(\omega)\, d\omega = \int_{-\infty}^{\infty} \varphi(\omega)\, m(d\omega)$ for all test functions φ. A version of the Kakutani–Riesz representation theorem implies that a distribution P is a positive measure if and only if $\int_{-\infty}^{\infty} P(\omega)\, \varphi(\omega)\, d\omega \geqslant 0$ for every nonnegative test function φ; (cf. ref. 4, Theorem 2.1.7).

To see now how one might define the power spectrum as a distribution, consider the *finite-time power spectrum* defined by

$$P_T(\omega) = \frac{1}{T} \left| \int_0^T e^{-i\omega t} g(t)\, dt \right|^2 \qquad \text{for} \quad T > 0 \tag{3.1.2}$$

and the *finite-time autocorrelation*

$$C_T(\tau) = \frac{1}{T} \int_0^T g(t+\tau)\, g(t)\, dt \qquad \text{for} \quad \tau \in (-T, \infty)$$

of g. Let φ be a test function, and let

$$\hat{\varphi}(\tau) = \int_{-\infty}^{\infty} e^{i\tau\omega} \varphi(\omega)\, d\omega$$

be its Fourier transform. Observe that

$$\begin{aligned}
&\int_{-\infty}^{\infty} P_T(\omega)\, \varphi(\omega)\, d\omega - \int_{-T}^{T} C_T(\tau)\, \hat{\varphi}(\tau)\, d\tau \\
&\quad = \frac{1}{T} \int_0^T \int_0^T g(t_1)\, g(t_2)\, dt_1\, dt_2 \int_{-\infty}^{\infty} e^{i\omega(t_2 - t_1)} \varphi(\omega)\, d\omega \\
&\qquad - \frac{1}{T} \int_0^T g(t)\, dt \int_{-T}^{T} g(t+\tau)\, \hat{\varphi}(\tau)\, d\tau \\
&\quad = \frac{1}{T} \int_0^T \int_0^T g(t_1)\, g(t_2)\, \hat{\varphi}(t_2 - t_1)\, dt_1\, dt_2 \\
&\qquad - \frac{1}{T} \int_0^T g(t_1)\, dt_1 \int_{-T+t_1}^{T+t_1} g(t_2)\, \hat{\varphi}(t_2 - t_1)\, dt_2 \\
&\quad = -\frac{1}{T} \int_0^T g(t_1)\, dt_1 \left(\int_T^{T+t_1} + \int_{-T+t_1}^{0} \right) g(t_2)\, \hat{\varphi}(t_2 - t_1)\, dt_2
\end{aligned}$$

whence

$$\left|\int_{-\infty}^{\infty} P_T(\omega)\,\varphi(\omega)\,d\omega - \int_{-T}^{T} C_T(\tau)\,\hat{\varphi}(\tau)\,d\tau\right|$$
$$\leqslant \frac{1}{T}(\sup_t |g|)^2 \int_0^T dt_1 \left(\int_T^{T+t_1} + \int_{-T+t_1}^{0}\right) |\hat{\varphi}(t_2 - t_1)|\,dt_2$$
$$\leqslant \frac{1}{T}(\sup_t |g|)^2 \int_{-T}^{T} |t\hat{\varphi}(t)|\,dt$$
$$\leqslant \frac{1}{T}(\sup_t |g|)^2 \int_{-\infty}^{\infty} |t\hat{\varphi}(t)|\,dt$$

so that

$$\int_{-\infty}^{\infty} P_T(\omega)\,\varphi(\omega)\,d\omega - \int_{-T}^{T} C_T(\tau)\,\hat{\varphi}(\tau)\,d\tau = O\left(\frac{1}{T}\right) \quad \text{for} \quad T \to \infty \qquad (3.1.3)$$

Now, for instance, if the autocorrelation of g

$$\lim_{T\to\infty} C_T(\tau) = C(\tau) \qquad \text{for} \quad \tau \in (-\infty, \infty) \qquad (3.1.4)$$

exists almost everywhere, then (3.1.3) (combined with the dominated convergence theorem; cf. ref. 8, Theorem 1.34) shows that $P(\omega) = \lim_{T\to\infty} P_T(\omega)$ exists in $\mathscr{D}'(\mathbb{R})$ (i.e., as a distribution in the sense of L. Schwartz) and is precisely the Fourier transform (again in the sense of Schwartz) of $C(\tau)$; cf. Section 7.1 in ref. 4. Indeed, one must only note that the finite-time autocorrelation functions are uniformly bounded by the upper bound of $|g|$. We summarize this discussion as follows.

3.2. *Proposition.* Let g be a bounded, measurable, real-valued function on $\mathbb{R}$, and assume that the autocorrelation C of g exists almost everywhere. Then the power spectrum P of g exists as a distribution, the Fourier transform of C is P, and in addition P is a positive measure.

Proof. We only have to verify that $\int_{-\infty}^{\infty} P(\omega)\,\varphi(\omega)\,d\omega \geqslant 0$ if φ is a nonnegative test function. This, however, is immediate because $P_T \geqslant 0$ and $\int_{-\infty}^{\infty} P(\omega)\,\varphi(\omega)\,d\omega = \lim_{T\to\infty} \int_{-\infty}^{\infty} P_T(\omega)\,\varphi(\omega)\,d\omega$.

As an example, consider the constant function $g(t) \equiv g_0$ for which $C(\tau) \equiv g_0^2$. We conclude that P is the Fourier transform of the constant function g_0^2, that is, $P = 2\pi g_0^2 \delta$, where δ is the Dirac "function."

A version of Proposition 3.2 can be proved even when the autocorrelation of g does not exist. In order to do that, we fix a functional Lim which extends the ordinary limit to the Banach space $B(0, \infty)$ of all bounded functions on $(0, \infty)$. More precisely, we want that $|\mathrm{Lim}_{T\to\infty} f(T)| \leqslant \|f\|_\infty$ and $\mathrm{Lim}_{T\to\infty} f(T) = \lim_{T\to\infty} f(T)$ if $f\in B(0, \infty)$ has a limit at infinity. The existence of such functionals is an easy consequence of the Hahn–Banach theorem; cf. ref. 3, Theorem II.3.10. Any such functional Lim is positive, in the sense that the Lim of a nonnegative function is nonnegative.

3.3. *Proposition.* Let g be a bounded function. For every test function φ the function $T\mapsto \int_{-\infty}^{\infty} P_T(\omega)\, \varphi(\omega)\, d\omega$ is bounded, and the formula

$$\int_{-\infty}^{\infty} P(\omega)\, \varphi(\omega)\, d\omega = \mathop{\mathrm{Lim}}_{T\to\infty} \int_{-\infty}^{\infty} P_T(\omega)\, \varphi(\omega)\, d\omega$$

defines a distribution. The distribution P thus defined is a positive measure.

Proof. Relation (3.1.3) implies the boundedness of $T\mapsto \int_{-\infty}^{\infty} P_T(\omega)\, \varphi(\omega)\, d\omega$, as well as the fact that

$$\mathop{\mathrm{Lim}}_{T\to\infty} \int_{-\infty}^{\infty} P_T(\omega)\, \varphi(\omega)\, d\omega = \mathop{\mathrm{Lim}}_{T\to\infty} \int_{-T}^{T} C_T(\tau)\, \hat{\varphi}(\tau)\, d\tau$$

Since the functions C_T are uniformly bounded, the existence of the distribution P follows. It is easily seen that $\int_{-\infty}^{\infty} P(\omega)\, \varphi(\omega)\, d\omega$ is nonnegative if $\varphi \geqslant 0$.

Since the dominated convergence theorem does not apply to Lim, we cannot generally conclude that the distribution P defined in Proposition 3.2 is the Fourier transform of the bounded function $C(\tau) = \mathrm{Lim}_{T\to\infty} C_T(\tau)$. This, however, can be proved under additional assumptions on the function g.

3.4. *Proposition.* Assume that the function g is bounded and uniformly continuous on $(-\infty, \infty)$. Then the distribution P defined in Proposition 3.2 is the Fourier transform of the bounded function $C(\tau) = \mathrm{Lim}_{T\to\infty} C_T(\tau)$.

Proof. We must show that

$$\mathop{\mathrm{Lim}}_{T\to\infty} \int_{-T}^{T} C_T(\tau)\, \hat{\varphi}(\tau)\, d\tau = \int_{-\infty}^{\infty} C(\tau)\, \hat{\varphi}(\tau)\, d\tau$$

for every $\varphi \in \mathscr{D}(\mathbb{R})$. We will prove that

$$\mathop{\mathrm{Lim}}_{T\to\infty} \int_{-T}^{T} C_T(\tau)\, f(\tau)\, d\tau = \int_{-\infty}^{\infty} C(\tau)\, f(\tau)\, d\tau$$

for every $f\in L^1(\mathbb{R})$. Now, the functions $C_T(\tau)$ are uniformly bounded, so that it suffices to prove the identity above for functions f with compact support. Assume therefore that $f \in L^1(\mathbb{R})$ is supported in the interval $[-T_0, T_0]$. For each $\tau \in [-T_0, T_0]$ define a function $\xi_\tau \in B(0, \infty)$ by

$$\xi_\tau(T) = \begin{cases} 0 & \text{if } \; T < T_0 + 1 \\ C_T(\tau) & \text{if } \; T \geqslant T_0 + 1 \end{cases}$$

Clearly we have

$$\operatorname*{Lim}_{T \to \infty} \xi_\tau(T) = \operatorname*{Lim}_{T \to \infty} C_T(\tau) \qquad \text{for} \quad \tau \in [-T_0, T_0]$$

We claim that the map $\tau \mapsto \xi_\tau$ is continuous from $[-T_0, T_0]$ to $B(0, \infty)$. Indeed, if $\tau, \tau' \in [-T_0, T_0]$ and $|\tau - \tau'| \leqslant 1$,

$$\begin{aligned} \|\xi_\tau - \xi_{\tau'}\|_\infty &= \sup_{T \geqslant 0} |\xi_\tau(T) - \xi_{\tau'}(T)| \\ &= \sup_{T \geqslant T_0 + 1} |\xi_\tau(T) - \xi_{\tau'}(T)| \\ &= \sup_{T \geqslant T_0 + 1} |C_T(\tau) - C_T(\tau')| \\ &\leqslant \sup_{T \geqslant T_0 + 1} \frac{1}{T} \int_0^T |g(t+\tau) - g(t+\tau')| \cdot |g(t)| \, dt \\ &\leqslant \|g\|_\infty \sup_{t \geqslant 0} |g(t+\tau) - g(t+\tau')| \end{aligned}$$

and this tends to zero as $|\tau - \tau'| \to 0$ by the uniform continuity of g. It follows that the function $f(\tau)\, \xi_\tau$ is an integrable $B(0, \infty)$-valued function, and the identity to be proved can be rewritten as

$$\operatorname*{Lim}_{T \to \infty} \int_{-\infty}^{\infty} f(\tau)\, \xi_\tau \, d\tau = \int_{-\infty}^{\infty} f(\tau) \operatorname*{Lim}_{T \to \infty} \xi_\tau \, d\tau$$

This identity simply follows from the continuity of the functional $\operatorname{Lim}_{T \to \infty}$.

It is important to observe that the power spectrum and autocorrelation of a function g only depend on the values $g(t)$ for large t. To be more precise, we formulate this as a separate result.

3.5. *Proposition.* Let g_1 and g_2 be two bounded measurable functions, and denote by P_T^1, P_T^2, C_T^1, and C_T^2 the corresponding finite-time power spectra and autocorrelations. If $g_1(t) = g_2(t)$ for sufficiently large t, then:

(1) $\operatorname*{Lim}_{T \to \infty} C_T^1(\tau) = \operatorname*{Lim}_{T \to \infty} C_T^2(\tau) \qquad$ for all τ

(2) $\displaystyle \operatorname*{Lim}_{T\to\infty}\int_{-T}^{T} C_T^1(\tau)\,\hat{\varphi}(\tau)\,d\tau=\operatorname*{Lim}_{T\to\infty}\int_{-T}^{T} C_T^2(\tau)\,\hat{\varphi}(\tau)\,d\tau$

for every $\varphi\in\mathscr{D}(\mathbb{R})$

(3) $\displaystyle \operatorname*{Lim}_{T\to\infty}\int_{-\infty}^{\infty} P_T^1(\omega)\,\varphi(\omega)\,d\omega=\operatorname*{Lim}_{T\to\infty}\int_{-\infty}^{\infty} P_T^2(\omega)\,\varphi(\omega)\,d\omega$

for every $\varphi\in\mathscr{D}(\mathbb{R})$

Proof. Suppose that $g_1(t)=g_2(t)$ for $t\geqslant T_0>0$, and A is a positive number. For $t>T_0+A$ and $|\tau|<A$ we have $g_1(t)=g_2(t)$ and $g_1(t+\tau)=g_2(t+\tau)$. It follows that $|C_T^1(\tau)-C_T^2(\tau)|\leqslant 2k^2(T_0+A)/T$, $\tau\in[-A,A]$, where k is a common bound for $|g_1|$ and $|g_2|$. The proposition follows immediately from this estimate.

Using the preceding result, one can define the power spectrum and autocorrelation for any bounded function g defined on an interval of the form $[T_0,+\infty)$. Indeed, one first chooses an arbitrary bounded extension of g to $\mathbb{R}$ and calculates the corresponding entities for this extension. If g is uniformly continuous on $[T_0,+\infty)$, the extension can be chosen uniformly continuous as well. We see in particular that the conclusion of Proposition 3.4 holds even if g is only assumed uniformly continuous on $[T_0,+\infty)$.

3.6. We are interested in the behavior of P for large values of ω. In view of the difficulties related to passing to the limit $T\to\infty$ and the fact that P may be a distribution even when those difficulties can be overcome, we will first convolve P_T with a standard mollifier and then study the limit for $T\to\infty$. To be precise, let ψ be a fixed smooth function which is even, nonnegative, and zero outside the interval $[-\omega_0,\omega_0]$, and $\int_{-\infty}^{\infty}\psi(\omega)\,d\omega=1$. We set

$$\psi_\varepsilon(\omega)=\frac{1}{\varepsilon}\psi\left(\frac{\omega}{\varepsilon}\right)\qquad\text{for}\quad \omega\in(-\infty,\infty),\quad \varepsilon>0 \tag{3.6.1}$$

and convolve P_T with ψ_ε:

$$P_{T,\varepsilon}(\omega)=\int_{-\infty}^{\infty} P_T(\sigma)\,\psi\left(\frac{\omega-\sigma}{\varepsilon}\right)\frac{d\sigma}{\varepsilon}=(P_T*\psi_\varepsilon)(\omega) \tag{3.6.2}$$

Clearly, $\lim_{\varepsilon\to\infty}P_{T,\varepsilon}=P_T$ uniformly, and if P exists as a distribution (e.g., the case considered in Section 3.1 above), we have $\lim_{T\to\infty}P_{T,\varepsilon}=P*\psi_\varepsilon$, which is a smooth function, even when P is a distribution; cf. Section 4.1 in ref. 4.

The object we want to estimate is

$$\limsup_{T\to\infty} |P_{T,\varepsilon}(\omega)|$$

for the case when $g(t) = u_j(t, x_0)$ $(j = 1, 2)$, where $u(t) = (u_1(t, x), u_2(t, x))$ $(x \in \Omega)$ is a solution of (2.1.1) on the global attractor $\mathscr{A}$, and x_0 is any fixed point in Ω.

3.7. *Proposition.* Let $P_T(\omega)$ be the finite-time power spectrum of a component $g(t) = u_j(t, x_0)$ of the velocity field at a point $x_0 \in \Omega$ of a solution $u(t)$ of (2.1.1) on the global attractor $\mathscr{A}$. Let, moreover, $P_{T,\varepsilon}$ be the convolution defined in (3.6.2). Then

$$\limsup_{T\to\infty} |P_{T,\varepsilon}(\omega)| \leqslant c_4^2 \frac{\beta_0^2}{\lambda_1}\left(4\delta_0 + \frac{\alpha}{\varepsilon} e^{-\delta_0(|\omega| - 2\varepsilon\omega_0)}\right) e^{-\delta_0(|\omega| - 2\varepsilon\omega_0)} \qquad (3.7.1)$$

for all $\omega \in (-\infty, \infty)$, $\varepsilon > 0$; in (3.7.1) α is a constant depending on the mollifier ψ.

This proposition provides a rigorous explanation of the fact that the power spectra of the velocity field at a fixed point in a 2D flow in an already permanent turbulent state are decaying exponentially for high frequencies. However, in a transient turbulent flow (i.e., when the solution is off the attractor $\mathscr{A}$), the conditions that prevail may suffice neither for the existence of P nor for the estimate (3.6.1).

3.8. *Proof of the Proposition.* According to the discussion in Section 2, if $u(t) \in \mathscr{A}$ is a solution on the attractor, then $u(t)$ can be extended in a band of the form $\{\zeta \in \mathbb{C}: |\Im\zeta| \leqslant \delta_0\}$ to an analytic function $U(\zeta, u_0)$ satisfying (2.4.1), where $u_0 = u(0)$. Now, for any $x_0 \in \Omega$ the map $v \mapsto v(x_0)$ is a linear continuous map from $\mathscr{D}(A)_{\mathbb{C}}$ into $\mathbb{C}$ satisfying also

$$\|v(x_0)\|_{\mathbb{C}^2} \leqslant c_4 |v|^{1/2} |Av|^{1/2} \leqslant c_4 \lambda_1^{-1/2} |Av| \qquad (3.8.1)$$

(see Agmon's inequality in ref. 2 or ref. 9). It follows that $u_j(t, x_0)$ has an analytic extension [the jth component of $U(\zeta, u_0)(x_0)$] to the whole band $\{\zeta \in \mathbb{C}: |\Im\zeta| \leqslant \delta_0\}$ which satisfies

$$|u_j(\zeta, x_0)| \leqslant c_4 \lambda_1^{-1/2} |AU(\zeta, u_0)| \leqslant c_4 \lambda_1^{-1/2} \beta_0$$

Relation (3.7.1) is now a direct consequence of the following result. Recall that the mollifier $\psi(\omega)$ was supposed to be zero for $|\omega| > \omega_0$.

3.9. *Lemma.* Let g be a bounded analytic function in the strip $\{\zeta \in \mathbb{C}: |\Im\zeta| \leqslant \delta_0\}$ with real values on the real axis. Let $P_T(\omega)$ be the finite-time power spectrum of g and let $P_{T,\varepsilon} = P_T * \psi$ be defined as in (3.6.2). Then

$$\limsup_{T \to \infty} |P_{T,\varepsilon}(\omega)| \leqslant \left(\sup_{|\Im\zeta| \leqslant \delta_0} |g(\zeta)|\right)^2 \left(4\delta_0 + \frac{\alpha}{\varepsilon} e^{-\delta_0(|\omega| - 2\varepsilon\omega_0)}\right) e^{-\delta_0(|\omega| - 2\varepsilon\omega_0)} \tag{3.9.1}$$

for all $\omega \in (-\infty, \infty)$, $\varepsilon > 0$, where α is a constant depending on ψ and satisfies

$$\alpha \geqslant e^{-\delta_0\omega_0} \int_{-\infty}^{\infty} |\hat{\psi}(\tau \pm 2i\delta_0)| \, d\tau \tag{3.9.2}$$

Proof. We will prove the lemma for positive values of ω. The proof for $\omega < 0$ is entirely analogous and will therefore be omitted. Let

$$g_T(\omega) = \int_0^T e^{-i\omega t} g(t) \, dt \qquad \text{for} \quad \omega \in (-\infty, \infty), \quad T > 0$$

and take $\omega \geqslant 0$. The Cauchy theorem (cf. ref. 8, Theorem 10.12) in the rectangle $\{x + iy\colon -\delta \leqslant y \leqslant 0,\ 0 \leqslant x \leqslant T\}$, with $0 < \delta \leqslant \delta_0$, implies that

$$\begin{aligned} \left| g_T(\omega) - \int_0^T e^{-i(t - i\delta)\omega} g(t - i\delta) \, dt \right| &\leqslant \left(\sup_{-\delta \leqslant s \leqslant 0} |g(is)|\right) \delta + \left(\sup_{-\delta \leqslant s \leqslant 0} |g(T + is)|\right) \delta \\ &\leqslant 2\gamma\delta \end{aligned}$$

with $\gamma = \sup_{|\Im\zeta| \leqslant \delta} |g(\zeta)|$. Thus we can write $g_T = h_T + k_T$, where $h_T(\omega) = \int_0^T e^{-i\omega(t - i\delta)} g(t - i\delta) \, dt$ and $|k_T(\omega)| \leqslant 2\gamma\delta$. Therefore

$$P_T(\omega) = \frac{1}{T} |g_T(\omega)|^2 = \frac{1}{T} |h_T(\omega)|^2 + 2\Re \frac{1}{T} h_T(\omega) \overline{k_T(\omega)} + \frac{1}{T} |k_T(\omega)|^2$$

and hence

$$P_{T,\varepsilon} \leqslant \frac{1}{T} |h_T|^2 * \psi_\varepsilon + \frac{4\gamma\delta}{T} |h_T| * \psi_\varepsilon + \frac{4\gamma^2\delta^2}{T} 1 * \psi_\varepsilon$$

Since $1 * \psi_\varepsilon(\omega) = 1$ for all ω, and

$$\frac{4\gamma\delta}{T} (|h_T| * \psi_\varepsilon)(\omega) \leqslant 4\gamma^2 \delta e^{-\omega\delta}$$

it follows that

$$\limsup_{T\to\infty} P_{T,\varepsilon}(\omega) \leqslant \limsup_{T\to\infty} \frac{1}{T}(|h_T|^2 * \psi_\varepsilon)(\omega) + 4\gamma^2\delta e^{-\omega\delta} \tag{3.9.3}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{T}(|h_T|^2 * \psi_\varepsilon)(\omega)\\
&\quad = \frac{1}{T}\int_0^T\int_0^T g(t-i\delta)\,\overline{g(s-i\delta)}\,ds\,dt \int_{-\infty}^{\infty} e^{-i\sigma(t-s-2i\delta)}\frac{1}{\varepsilon}\psi\left(\frac{\omega-\sigma}{\varepsilon}\right)d\sigma\\
&\quad = \frac{1}{T}\int_0^T\int_0^\infty g(t-i\delta)\,\overline{g(s-i\delta)}\,e^{-i\omega(t-s-2i\delta)}\hat{\psi}(\varepsilon(t-s-2i\delta))\,ds\,dt
\end{aligned}$$

Since $|g(t-i\delta)| \leqslant \gamma$ and $|g(s-i\delta)| \leqslant \gamma$, we obtain the estimate

$$\begin{aligned}
&\frac{1}{T}(|h_T|^2 * \psi_\varepsilon)(\omega)\\
&\quad = \frac{\gamma^2}{T}e^{-2\delta\omega}\int_0^T\int_0^T |\hat{\psi}(\varepsilon(t-s-2i\delta))|\,ds\,dt\\
&\quad \leqslant \gamma^2 e^{-2\delta\omega}\int_{-T}^{T} |\hat{\psi}(\varepsilon(t-2i\delta))|\,dt \leqslant \gamma^2 e^{-2\delta\omega}\int_{-\infty}^{\infty} |\hat{\psi}(\varepsilon(t-2i\delta))|\,dt
\end{aligned}$$

From (3.9.3) we can now infer

$$\limsup_{T\to\infty} P_{T,\varepsilon}(\omega) \leqslant \gamma^2(\alpha_\varepsilon e^{-\delta\omega} + 4\delta)\,e^{-\delta\omega} \qquad \text{for all} \quad \omega \geqslant 0 \tag{3.9.4}$$

where

$$\alpha_\varepsilon = \int_{-\infty}^{\infty} |\hat{\psi}(\varepsilon(t-2i\delta))|\,dt \tag{3.9.5}$$

Moreover, it is clear that

$$|\hat{\psi}(\zeta)| \leqslant \gamma_1(1+|\zeta|^2)^{-1}\,e^{\omega_0|\Im\zeta|} \qquad \text{for all} \quad \zeta \in \mathbb{C}^2$$

with an appropriate constant γ_1 depending only on ψ. This yields the estimate

$$\alpha_\varepsilon \leqslant \frac{\gamma_1}{\varepsilon}\,e^{2\varepsilon\delta\omega_0}\pi$$

Setting $\alpha = \pi\gamma_1$, $\delta = \delta_0$, we obtain (3.9.1) for $\omega \geqslant 0$. This concludes the proof of the lemma and hence that of Proposition 3.7.

4. THE STATISTICAL POWER SPECTRUM

4.1. In Section 3.1 we discussed the difficulties arising from the definition (3.1.1) of the power spectrum. For the special case of interest, namely that considered in Section 3.7, there are two ways to overcome those difficulties. The first way is to refer directly to the statistics provided by an invariant probability measure with respect to $\{S(t)\}_{t\geqslant 0}$ and to invoke ergodic theory. We will first deal with this approach, while the second is treated in Section 5.

Recall that an *invariant probability measure* μ for the semigroup $\{S(t)\}_{t\geqslant 0}$ is a probability measure defined on the Borel subsets of H with the property that

$$\mu(S(t)^{-1}\mathscr{B})=\mu(\mathscr{B}) \qquad \text{for all} \quad t\geqslant 0 \quad \text{and all Borel sets} \quad \mathscr{B}\subset H \tag{4.1.1}$$

Invariance is equivalent to the equality

$$\int \Phi(S(t)\,u)\,\mu(du)=\int \Phi(u)\,\mu(du) \qquad \text{for} \quad t\geqslant 0 \tag{4.1.2}$$

for all integrable real-valued Borel functions on H. Invariant measures for $\{S(t)\}_{t\geqslant 0}$ exist, and Proposition 5.3 below provides a construction of such measures.

4.2. *Lemma.* Any invariant probability measure μ is carried by the global attractor $\mathscr{A}$, that is, $\mu(H\backslash\mathscr{A})=0$.

Proof. First let B_a be a closed ball in $\mathscr{D}(A)$ which is absorbing for $\{S(t)\}_{t\geqslant 0}$ (see the remark at the end of Section 2.2) and let $B(r)=\{u\in H: |u|\leqslant r\}$ be some (closed) ball in H. There exists a time $t>0$ such that $B(r)\subset S(t)^{-1}B_a$. Thus

$$1\geqslant\mu(B_a)=\mu(S(t)^{-1}B_a)\geqslant\mu(B(r))\rightarrow\mu(H)=1 \qquad \text{for} \quad r\rightarrow\infty$$

and therefore $\mu(B_a)=1$. Let t_a be a time such that $S(t)\,B_a\subset B_a$ for $t\geqslant t_a$. Then $S(kt_a)\,B_a\supset S((k+1)\,t_a)\,B_a$ for all $k=0, 1, 2,\ldots$, and

$$\mathscr{A}=\bigcap_{k=0}^{\infty} S(kt_a)\,B_a$$

Therefore

$$1\geqslant\mu(\mathscr{A})=\lim_{k\rightarrow\infty}\mu(S(kt_a)\,B_a)=\lim_{k\rightarrow\infty}\mu(S(kt_0)^{-1}\,S(kt_0)\,B_a)=\mu(B_a)=1$$

which concludes the proof.

4.3. By virtue of Lemma 4.2, in studying invariant probability measures we can restrict our considerations to $\mathscr{A}$, and in particular work with Borel functions which are only defined on $\mathscr{A}$. In our further discussion we will systematically use the functions δ_{jx_0} on $\mathscr{D}(A)_{\mathbb{C}}$ defined by $\delta_{jx_0}(v) = v_j(x_0)$, $v \in \mathscr{D}(A)_{\mathbb{C}}$, $x_0 \in \Omega$, $j = 1, 2$. These functions have already proved useful in Section 3.8. Since δ_{jx_0} is linear and continuous on $\mathscr{D}(A)_{\mathbb{C}}$ [see (3.8.1)], the function

$$\Psi_{j,x_0,\tau}(u) = \delta_{jx_0}(S_{\mathscr{A}}(\tau)\, u)\, \delta_{jx_0}(u), \qquad u \in \mathscr{A} \tag{4.3.1}$$

is continuous, and hence bounded on $\mathscr{A}$. Moreover, these functions enjoy the following remarkable property:

$$|\Psi_{j,x_0,\tau_1}(u) - \Psi_{j,x_0,\tau_2}(u)| \leqslant c_4^2 \lambda_1^{-1} \beta_0^2 \delta_0^{-1}\, |\tau_1 - \tau_2| \tag{4.3.2}$$

for all real τ_1, τ_2, all $u \in \mathscr{A}$, and all $x_0 \in \Omega$.

Proof of (4.3.2). We have

$$\begin{aligned}
&|\Psi_{j,x_0,\tau_1}(u) - \Psi_{j,x_0,\tau_2}(u)| \\
&\quad = |(S_{\mathscr{A}}(\tau_1)\, u)_j\, (x_0)\, u_j(x_0) - (S_{\mathscr{A}}(\tau_2)\, u)_j\, (x_0)\, u_j(x_0)| \\
&\quad \leqslant |U(\tau_1, u)_j\, (x_0) - U(\tau_2, u)_j\, (x_0)| \cdot |u_j(x_0)| \\
&\quad \leqslant c_4^2 \lambda_1^{-1}\, |AU(\tau_1, u) - AU(\tau_2, u)| \cdot |Au| \\
&\quad \leqslant c_4^2 \lambda_1^{-1}\, |\tau_1 - \tau_2| \left(\max_{-\infty < \tau < \infty} \left| A \frac{d}{d\tau} U(\tau, u) \right| \right) |Au| \\
&\quad \leqslant c_4^2 \lambda_1^{-1}\, |\tau_1 - \tau_2|\, \beta_0 \left(\max_{-\infty < \tau < \infty} \left| A \frac{d}{d\tau} U(\tau, u) \right| \right)
\end{aligned}$$

where we used (3.8.1) in the second inequality and (2.4.1) in the last inequality. We use now the fact that the function $U(\zeta, u)$ is a $\mathscr{D}(A)_{\mathbb{C}}$-valued analytic function on $\{\zeta \in \mathbb{C}\colon |\Im\zeta| \leqslant \delta_0\}$, in particular, on the disk $\{\zeta \in \mathbb{C}\colon |\zeta - \tau| \leqslant \delta_0\}$ for every $\tau \in \mathbb{R}$. The Cauchy inequality for vector-valued functions applied in these disks, along with (2.4.1), implies

$$\left| A \frac{d}{dt} U(\tau, u) \right| \leqslant \frac{1}{\delta_0} \max_{|\zeta - \tau| \leqslant \delta_0} |AU(\zeta, v)| \leqslant \frac{\beta_0}{\delta_0}$$

Relation (4.3.2) now follows immediately from this and the estimate obtained earlier.

4.4. *Proposition.* Let μ be an invariant probability measure and let $x_0 \in \Omega$ and $j = 1, 2$ be fixed. Then for all μ-almost every $u \in \mathscr{A}$ the autocorrelation

$$C(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T (S_{\mathscr{A}}(t+\tau)\, u)_j\, (x_0)(S_{\mathscr{A}}(t)\, u)_j\, (x_0)\, dt \qquad (4.4.1)$$

exists for all $\tau \in \mathbb{R}$. Moreover, the power spectrum $P(\omega)$ of the function $g(t) = u_j(t, x_0) = (S_{\mathscr{A}}(t)\, u)_j\, (x_0)$ exists as a distribution, and it is the Fourier transform of $C(\tau)$ in the sense of L. Schwartz. The distribution $P(\omega)$ is in fact a positive Borel measure.

Proof. By virtue of (2.4.1) and (3.8.1) the function g is bounded. Thus, by Proposition 3.2, it suffices to prove the existence of the autocorrelation. Let $\mathbb{Q}$ denote the subset of all rational numbers in $\mathbb{R}$. Birkhoff's ergodic theorem (see ref. 3, Theorem VIII.6.12) implies the existence for each $\tau \in \mathbb{Q}$ of a subset $\mathscr{A}_\tau \subset \mathscr{A}$ of μ-measure zero such that

$$\begin{aligned} C(\tau) &= \lim_{T \to \infty} \frac{1}{T} \int_0^T u_j(t+\tau, x_0)\, u_j(t, x_0)\, dt \\ &= \lim_{T \to \infty} \frac{1}{T} \int_0^T \Psi_{j, x_0, \tau}(S(t)\, u)\, dt \end{aligned} \qquad (4.4.2)$$

exists for all $u \in \mathscr{A} \backslash \mathscr{A}_\tau$. Since $\mathbb{Q}$ is countable, the set $\mathscr{E} = \bigcup_{\tau \in Q} \mathscr{A}_\tau$ also has μ-measure zero, and for $u \in \mathscr{A} \backslash \mathscr{E}$ the limit $C(\tau)$ as defined in (4.4.2) exists for all $\tau \in \mathbb{Q}$. We will show that for $u \in \mathscr{A} \backslash \mathscr{E}$ the limit $C(\tau)$ exists for every $\tau \in \mathbb{R}$. Indeed, let $u \in \mathscr{A} \backslash \mathscr{E}$ and $\tau \in \mathbb{R}$ be arbitrary. For $\varepsilon > 0$ choose $\tau_\varepsilon \in Q$ such that $|\tau_\varepsilon - \tau| \leqslant \varepsilon$. Then, by virtue of (4.3.2), we have

$$\left| \frac{1}{T} \int_0^T \Psi_{j, x_0, \tau}(S_{\mathscr{A}}(t)\, u)\, dt - \frac{1}{T} \int_0^T \Psi_{j, x_0, \tau_\varepsilon}(S_{\mathscr{A}}(t)\, u)\, dt \right| \leqslant c_4^2 \lambda_1^{-1} \beta_0^2 \delta_0^{-1} \varepsilon$$

Since $C(\tau_\varepsilon)$ exists, we infer that

$$\begin{aligned} &\limsup_{T \to \infty} \frac{1}{T} \int_0^T \Psi_{j, x_0, \tau}(S_{\mathscr{A}}(t)\, u)\, dt - \liminf_{T \to \infty} \frac{1}{T} \int_0^T \Psi_{j, x_0, \tau}(S_{\mathscr{A}}(t)\, u)\, dt \\ &\quad \leqslant c_1^2 \lambda_1^{-1} \beta_0^2 \delta_0^{-1} \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $C(\tau)$ exists, as claimed. This concludes the proof.

4.5. *Remark.* With the notation of Section 3.7, when (4.4.1) exists for all $\tau \in \mathbb{R}$ we have

$$\begin{aligned}\left|\int_{-\infty}^{\infty} \frac{1}{\varepsilon}\psi\left(\frac{\omega-\sigma}{\varepsilon}\right) m(d\sigma)\right| &= |(P * \psi_\varepsilon)(\omega)| \\ &\leqslant \lambda_1^{-1}\beta_0^2\left(4\delta_0+\frac{\alpha}{\varepsilon}e^{-\delta_0(|\omega|-2\varepsilon\omega_0)}\right)e^{-\delta_0(|\omega|-2\varepsilon\omega_0)} \quad \text{for} \quad \omega\in\mathbb{R}, \quad \varepsilon>0\end{aligned} \tag{4.5.1}$$

In the next section we give an integral version of this exponential decay property which is independent of ε.

5. GENERAL EXISTENCE THEOREM

5.1. There are still some problems concerning the existence of the power spectrum of the velocity field at a point x_0 in Ω which we have not yet addressed. First, how rich is the family of all invariant probability measures? Second, what happens if the velocity field is not on the global attractor? In this section we will give *one* answer to both of these problems, as well as the supplement to (4.5.1) promised in Section 4.5.

5.2. In order to keep the notation as near as possible to that widely used in physics, and also to overcome the difficulty that in a rigorous mathematical study we cannot expect that all time averages of the form

$$\frac{1}{T}\int_0^T \Phi(S(t)\,u)\,dt$$

have limits for $T\to\infty$, we will work with an extension Lim of the ordinary limit, as in Proposition 3.3.

5.3. *Proposition.* For every $u_0\in\mathscr{D}(A)$ there exists an invariant probability measure μ for $\{S(t)\}_{t\geqslant 0}$ such that

$$\operatorname*{Lim}_{T\to\infty}\frac{1}{T}\int_0^T \Phi(S(t)\,u_0)\,dt = \int \Phi(u)\,\mu(du) \tag{5.3.1}$$

for all continuous functions Φ on $\mathscr{D}(A)$.

Proof. We consider again an absorbing set B_a which is a closed ball in $\mathscr{D}(A)$, and a time t_a such that $S(t)\,B_a\subset B_a$ for all $t\geqslant t_a$. It is easy to check that the absorbing set $X=S(t_a)\,B_a$ is also compact in $\mathscr{D}(A)$. Let

$C(X)$ denote the space of all real continuous functions on X [with respect to the norm of $\mathscr{D}(A)$]. Let Φ be a continuous function on $\mathscr{D}(A)$. Since $S(t)\,u_0 \in X$ for sufficiently large t, we conclude that the function $t \mapsto \Phi(S(t)\,u_0)$ is bounded, and hence we can define

$$G(\Phi) = \operatorname*{Lim}_{T \to \infty} \frac{1}{T} \int_0^T \Phi(S(t)\,u_0)\,dt$$

We claim that $G(\Phi)$ only depends on the restriction $\Phi \mid X$. Indeed, if $\Phi \mid X = 0$, then $\Phi(S(t))\,u_0 = 0$ for large values of T, and clearly the limit of $(1/T) \int_0^T \Phi(S(t)\,u_0)\,dt$ is zero, in particular, $G(\Phi) = 0$. By Tietze's theorem (cf. Theorem I.5.3 in ref. 3), any function $\Psi \in C(X)$ can be extended to a continuous function on $\mathscr{D}(A)$. The previous argument allows us to define a functional F on $C(X)$ by setting $F(\Psi) = G(\Phi)$, where Φ is an arbitrary continuous extension of Ψ to $\mathscr{D}(A)$. Clearly F is linear and positive in the sense that $F(\Psi) \geqslant 0$ if $\Psi \geqslant 0$. Thus by virtue of the Kakutani–Riesz representation theorem (cf. Theorem 2.14 in ref. 8) there exists a positive (Borel) measure μ on X (and hence on H) such that

$$F(\Psi) = \int_X \Psi(u)\,\mu(du) \qquad \text{for} \quad \Psi \in C(X) \tag{5.3.2}$$

Since

$$\mu(X) = F(1) = \operatorname*{Lim}_{T \to \infty} 1 = 1$$

μ is a probability measure. Clearly μ satisfies (5.3.1) for all continuous functions Φ on $\mathscr{D}(A)$. It remains to show that μ is invariant. Indeed, if Φ is continuous on $\mathscr{D}(A)$ and $\tau > 0$, we have

$$\begin{aligned}
&\int \Phi(S(\tau)\,u)\,\mu(du) \\
&\quad = \operatorname*{Lim}_{T \to \infty} \frac{1}{T} \int_0^T \Phi(S(t+\tau)\,u_0)\,dt \\
&\quad = \operatorname*{Lim}_{T \to \infty} \frac{1}{T} \int_\tau^{T+\tau} \Phi(S(t)\,u_0)\,dt \\
&\quad = \operatorname*{Lim}_{T \to \infty} \frac{1}{T} \int_0^T \Phi(S(t)\,u_0)\,dt \\
&\qquad + \operatorname*{Lim}_{T \to \infty} \left\{ \frac{1}{T} \int_T^{T+\tau} \Phi(S(t+\tau)\,u_0)\,dt - \frac{1}{T} \int_0^\tau \Phi(S(t)\,u_0)\,dt \right\}
\end{aligned}$$

$$= \int \Phi(u)\, d\mu(u)$$

$$+ \lim_{T \to \infty} \left\{ \frac{1}{T} \int_T^{T+\tau} \Phi(S(t+\tau)\, u_0)\, dt - \frac{1}{T} \int_0^\tau \Phi(S(t)\, u_0)\, dt \right\}$$

$$= \int \Phi(u)\, d\mu(u)$$

This concludes our proof.

This proposition is an adaptation of a classical argument of Krylov and Bogoliubov[5] for the construction of invariant measures. As we will see below, the proposition is also a useful tool to replace the ad hoc assumption of ergodicity for the invariant probability measures.

5.4. *Proposition.* Fix $u_0 \in \mathscr{D}(A)$, $x_0 \in \Omega$, $j = 1, 2$, and set

$$g(t) = \begin{cases} (S(t)\, u)_j\, (x_0) & \text{if } \ t \geqslant 0 \\ 0 & \text{if } \ t < 0 \end{cases} \tag{5.4.1}$$

(1) For every $\phi \in \mathscr{D}(\mathbb{R})$, the function $T \mapsto \int_{-\infty}^{\infty} P_T(\omega)\, \phi(\omega)\, d\omega$ is bounded, and the formula

$$\int_{-\infty}^{\infty} P(\omega)\, \phi(\omega)\, d\omega = \operatorname*{Lim}_{T \to \infty} \int_{-\infty}^{\infty} P_T(\omega)\, \phi(\omega)\, d\omega \tag{5.4.2}$$

defines a distribution. The distribution P thus defined is a measure, i.e., $\int_{-\infty}^{\infty} P(\omega)\, \phi(\omega)\, d\omega = \int_{-\infty}^{\infty} \phi(\omega)\, m(d\omega)$, where m is a positive Borel measure on $\mathbb{R}$.

(2) The distribution P is the Fourier transform in the sense of Schwartz of the bounded function $C(\tau) = \operatorname{Lim}_{T \to \infty} C_T(\tau)$.

(3) The measure m satisfies the estimate

$$\int_{-\infty}^{\infty} (e^{\delta_0 \omega} + e^{-\delta_0 \omega})\, m(d\omega) \leqslant 2 c_4^2 \lambda_1^{-1} \beta_0^2 \tag{5.4.3}$$

Proof. Parts (1) and (2) follow from Propositions 3.3–3.5 and the remarks following Proposition 3.5, because $g(t)$ is indeed uniformly continuous for large values of t. The function $\Psi_{j, x_0, \tau}$ introduced in Section 4.3 is continuous on $\mathscr{D}(A)$. Therefore with the notation of Section 5.3 we have

$$C(\tau) = \operatorname*{Lim}_{T \to \infty} \frac{1}{T} \int_0^T \Psi_{j,x_0,\tau}(S(t)\, u_0)\, dt$$

$$= \int \Psi_{j,x_0,\tau}(u)\, \mu(du)$$

$$= \int \delta_{jx_0}(S(\tau)\, u)\, \delta_{jx_0}(u)\, \mu(du)$$

$$= \int \delta_{jx_0}(S_{\mathscr{A}}(\tau)\, u)\, \delta_{jx_0}(u)\, \mu(du) \tag{5.4.4}$$

for all $\tau \geqslant 0$. Noticing that

$$\int \delta_{jx_0}(S_{\mathscr{A}}(\tau)\, u)\, \delta_{jx_0}\, \mu(du) = \int \delta_{jx_0}(u)\, \delta_{jx_0}(S(-\tau)\, u)\, \mu(du)$$

and $C(\tau) = C(-\tau)$ for $\tau \leqslant 0$, we obtain

$$C(\tau) = \int_{\mathscr{A}} \delta_{jx_0}(S_{\mathscr{A}}(\tau)\, u)\, \delta_{jx_0}(u)\, \mu(du) \tag{5.4.5}$$

for $\tau \in \mathbb{R}$. Moreover, for $u \in \mathscr{A}$ and $\tau_0 \in \mathbb{R}$ we have

$$S_{\mathscr{A}}(\tau)\, u = U(\tau, u) = \sum_{n=0}^{\infty} (\tau - \tau_0)^n\, a_n(u, \tau_0) \qquad \text{for} \quad |\tau - \tau_0| < \delta_0$$

where by virtue of the vector version of the Cauchy inequalities we have

$$|Aa_n(u, t_0)| \leqslant \beta_0 / \delta_0^n \qquad (n = 0, 1, 2, \ldots)$$

Therefore

$$\delta_{jx_0}(S_{\mathscr{A}}(\tau)\, u) = \sum_{n=0}^{\infty} (\tau - \tau_0)^n\, \delta_{jx_0}(a_n(u, t_0)) \qquad \text{for} \quad |\tau - \tau_0| < \delta_0$$

where

$$|\delta_{jx_0}(a_n(u, t_0))| \leqslant c_4 \lambda_1^{-1/2} \beta_0 / \delta_0^n \qquad (n = 0, 1, 2, \ldots)$$

Thus

$$C(\tau) = \sum_{n=0}^{\infty} (\tau - \tau_0)^n\, \alpha_n \qquad \text{for} \quad |\tau - \tau_0| < \delta_0$$

where

$$\alpha_n = \int \delta_{jx_0}(a_n(u, t_0))\, \delta_{jx_0}(u)\, \mu(du)$$

$$|\alpha_n| \leqslant c_4^2 \lambda_1^{-1} \beta_0^2 / \delta_0^n \qquad \text{for} \quad n = 0, 1, 2,...$$

In particular,

$$\left| \left(\frac{d}{d\tau}\right) C(\tau) \right| \leqslant n!\, c_4^2 \lambda_1^{-1} \frac{\beta_0^2}{\delta_0^n} \qquad \text{for} \quad n = 0, 1, 2,... \tag{5.4.6}$$

But

$$-C''(0) = -\lim_{\tau \searrow 0} \frac{C(\tau) - 2C(0) + C(-\tau)}{\tau^2} = \lim_{\tau \searrow 0} \int_{-\infty}^{\infty} \left(\frac{\sin(\omega\tau/2)}{\tau/2} \right)^2 m(d\omega)$$

The Fatou Lemma (cf. Lemma 1.28 in ref. 8) implies that

$$\int_{-\infty}^{\infty} \omega^2 m(d\omega) \leqslant -C''(0) \tag{5.4.7}$$

Because of (5.4.7) we can write

$$-C''(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau} \omega^2 m(d\omega)$$

Repeating this argument, we find that

$$\int_{-\infty}^{\infty} \omega^{2n} m(d\omega) \leqslant (-1)^n\, C^{(2n)}(0), \qquad n = 0, 1, 2,... \tag{5.4.8}$$

and therefore, for $\delta \in [0, \delta_0)$, we have [by (5.4.8) and (5.4.6)]

$$\begin{aligned} &\int_{-\infty}^{\infty} \cosh(\delta\omega)\, m(d\omega) \\ &\quad = \sum_{n=0}^{\infty} \delta^{2n} \int_{-\infty}^{\infty} \frac{\omega^{2n}}{(2n)!} m(d\omega) \\ &\quad \leqslant \sum_{n=0}^{\infty} (-1)^n \frac{\delta^{2n}}{(2n)!} C^{(2n)}(0) = \sum_{n=0}^{\infty} \frac{(i\delta)^{2n}}{(2n)!} C^{(2n)}(0) \end{aligned}$$

But since C is even and analytic at 0 we have $C^{(n)}(0)=0$ if n is odd and therefore

$$\int_{-\infty}^{\infty} \cosh(\delta\omega)\, m(d\omega)$$

$$\leqslant \sum_{n=0}^{\infty} (i\delta)^{2n} \int \delta_{jx_0}(a_n(u,0))\, \delta_{jx_0}(u)\, \mu(du)$$

$$= \int \delta_{jx_0}(U(i\delta, u))\, \delta_{jx_0}(u)\, \mu(du) \leqslant c_4^2 \lambda_1^{-1} \beta_0^2$$

Obviously we can now let $\delta \nearrow \delta_0$. This concludes the proof of (3), and of the proposition.

5.5. *Remark.* Note that the relation (5.4.4) is satisfied by the autocorrelation $C(\tau)$ considered in Section 4. In particular, if $u_0 \in \mathscr{A} \backslash \mathscr{E}$ (see Sections 4.4 and 4.5), then applying Proposition 4.4 to u_0, we obtain condition (5.4.3), which supplements (4.5.1). Note that the invariant probability measure constructed in Section 5.3 and used in Section 5.4 in the proof of (5.4.3) may be different from the invariant probability measure considered *a priori* in Section 4.4, for which $\mathscr{E}$ is of zero probability.

The results in this section show that by replacing the classical operation $\lim_{T\to\infty}$ with the operation $\operatorname{Lim}_{T\to\infty}$ we can rigorously prove a set of results which were more or less intuitively or empirically known to physicists and engineers.

6. A REMARK ON THE THREE-DIMENSIONAL CASE

The difficulty in proving the results in Sections 3–5 for the 3D Navier–Stokes equations (with the same type of boundary conditions) lies in the fact that the existence of global (in time) regular solutions in this case is not yet known, and therefore the semigroup $\{S(t)\}_{t\geqslant 0}$ cannot be defined. However, at finite Reynolds numbers (which is always the case in our rigorous setting) the only way a solution can fail to be regular is if the velocity is infinite at some point in space and time. There is no experimental evidence even remotely suggesting such a situation. Assuming bounded velocity fields (in space and time), all the considerations made in the Sections 2–5 carry through to the three-dimensional case, with one caveat. Namely, we do not have any mathematical procedure for estimating the basic quantities δ_0 (the half-width of the strip of analyticity for the solutions on the global attractor) and β_0 (the bound of those solutions on the strip).

7. CONCLUSIONS

We have shown how the time analyticity of the solutions of the Navier–Stokes equations in two dimensions ensure the exponential dropoff of the spectrum of the temporal fluctuations of the turbulent velocity field. At the same time we found that the power spectrum is a measure and is not necessarily a classical function. From the practical point of view our considerations have led us to conclude that in this context the equality of time averages and ensemble averages can be obtained without invoking the assumption of the ergodic property for the turbulent velocity field.

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REFERENCES

1. E. Ching, P. Constantin, L. P. Kadanoff, A. Libchaber, I. Procaccia, and X.-Z. Wu, Transitions in convective turbulence: The role of thermal plumes, *Phys. Rev. A* **44**:8091–8102 (1991).
2. P. Constantin and C. Foias, *Navier–Stokes Equations* (University of Chicago Press, Chicago, 1988).
3. N. Dunford and J. T. Schwartz, *Linear Operators. Part I: General Theory* (Interscience, New York, 1958).
4. L. Hörmander, *The Analysis of Linear Partial Differential Operators* (Springer-Verlag, Berlin, 1983).
5. N. Krylov and N. N. Bogoliubov, La théorie générale de la mesure dans son application à l'étude des systèmes dinamiques de la mécanique non linèaire, *Ann. Math.* **38**:65–113 (1937).
6. Y. W. Lee, *Statistical Theory of Communication* (Wiley, New York, 1960).
7. J. Leray, Etude de divers équations intégrales non linéaires et de quelques problèmes que posent l'hydrodinamique, *J. Math. Pures Appl. 9e Ser.* **12**:1–82 (1933).
8. W. Rudin, *Real and Complex Analysis* (McGraw-Hill, New York, 1966).
9. R. Temam, *Navier–Stokes Equations: Theory and Numerical Analysis* (North-Holland, Amsterdam, 1977).
10. N. Wiener, *The Fourier Integral and Certain of Its Applications* (Dover, New York, 1958).
11. X.-Z. Wu, L. P. Kadanoff, A. Libchaber, and M. Sano, Frequency power spectrum of temperature fluctuations in free convection, *Phys. Rev. Lett.* **64**:2140–2143 (1990).